AN ESSAY

ON THE

Prevailing Systems of Instruction

IN

LEMENTARY MATHEMATICS

IN AN EDUCATIONAL CONVENTION HELD PURSUANT TO A RECOMMENDA-
TION OF THE GENERAL CONFERENCE OF THE METHODIST EPISCOPAL
CHURCH, SOUTH, IN THE CITY OF NASHVILLE, ON THE 22D OF
APRIL, 1856, AND PUBLISHED AT THE REQUEST OF THE
CONVENTION.

BY

JAMES B. DODD, A. M.,

MORRISON PROFESSOR OF MATHEMATICS, ETC., IN TRANSYLVANIA UNIVERSITY.

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ESSAY.

It may contribute towards the attainment of correct views of the present state of education in the department of elementary Mathematics, in our country, and the practicability of its amelioration, if we consider its present state in relation to its past, and, with a due sense of the responsibility assumed, endeavor to point out some of the defects, and their remedies, of the prevailing systems of instruction in these sciences.

Thirty years ago the *ipsi-dixit* method of Arithmetic was in vogue: the text-books in common use on this subject were on the high-authority principle: their rules were prescriptions which both school-master and school-boy must implicitly follow, without troubling themselves with the *reasons* on which those prescriptions were founded. It seems, moreover, not to have been expected that a majority of pupils would ever get through Arithmetic, or "cipher through the book," as it was called in those days; and, therefore, the most useful parts of Arithmetic, as they were esteemed—the Golden Rule of Three, for example —must come as early in the course as possible.

When Arithmetic was thus taught in a purely practical manner, without any theory whatever, and when one object seems to have been to teach the best parts of it first; it is not surprising that an arrangement should have been adopted, in the books, which is not the most friendly to a clear, consecutive, and scientific exposition of this subject as a whole. But it is sur-

prising that this arrangement, in some of its most objectionable features, should be retained in nearly all the Arithmetics of the present day, after so much industry has been expended in efforts to improve the character of text-books for schools.

In one respect, however, the character of school Arithmetics has been completely revolutionized. The old books were utterly wanting in necessary elementary matter, and in the exposition and application of *principles;* the new are very rich in explanations, preambles, inferences, analyses, demonstrations, etc. The old neither gave nor required any reason for any thing that was done: the new give and require reasons for every thing, and enjoin that not a figure shall be set down without its *why* and *wherefore.* The teaching according to the old books would not show the pupil why he must carry *one* for every *ten*, from right to left, in simple Addition; that according to the new will not allow him to extract the Cube Root until he has mastered the *rationale* of the Rule.

It is said that *extremes* sometimes meet. Every experienced and suceessful teacher knows that these extreme methods meet in the impossibility of ever managing Arithmetic, in schools, properly, by either of them. At the age at which this science must usually be studied, the pupil can comprehend a part of its theory, and he should be required to do so; other parts of it he cannot well comprehend, and he should be allowed to proceed with the practical operations, on the faith of the correctness of the Rules which are prescribed to him. At a more advanced stage in his mathematical studies, these omitted theories may be brought up in review; or should the pupil never get beyond the province of Arithmetic, in the business of life he may still walk by faith in the competency of those who have framed the practical rules of this science. The maxim in education that no advances must ever be made without a full comprehension of all the *principles* and *reasons* of every practical procedure, is an impracticable

one: it is contrary to the manner in which the largest portion of human knowledge ever has been, and ever must be, acquired.

In their efforts to make good the pretensious of this modern way of teaching, upon an unnatural arrangement of the subjects treated, the authors of our most popular Arithmetics have encumbered their works with a mass of verbiage which is really intolerable to any one who will discriminate the essential parts of this science—put them in logical order—and then invest them with just so much of precise and significant phraseology as is necessary to convey them into the understanding of the tyro. Let this work of rectification and reduction be completed on such Arithmetics as Davies', Ray's, and many others; and I speak knowingly when I say that the amount of expository matter necessary and proper to be laid before the pupil will be diminished *one-half*.

The study of Algebra has greatly extended, in our country, within twenty years past. From having been confined almost exclusively to the higher institutious, it has found its way into many of the Common Schools; so that it has become in a measure necessary that every teacher should know how to turn this key to the mathematical sciences. Bonnycastle's Algebra, improved by Ryan, was perhaps the most popular of the foreign treatises which preceded those of American authorship: Day's Algebra was, I think, the first of these: Davies' Bourdon, an abridgment of a French treatise, succeeded; and more recently have appeared Loomis's Algebra, and many others, adapted, professedly, to the usual College Course of Mathematics. But the desire of bringing this subject to some extent, within the capabilities of the younger classes of learners, has produced numerous introductory treatises, from which some of the essential, but reputedly more difficult parts of Algebra have been excluded.

The common practice with these two different orders of Alge-

braic treatises—and a similar course is pursued with the current *lower* and *higher* Arithmetics—is, to put the pupil first through the lower book, and then, for full instruction, to put him through the higher.

It is respectfully submitted that this is not the best way to manage these studies. It is not on the truly progressive plan; and must be attended with loss and perplexity in its practical working. Suppose the pupil to have mastered Davies' Common School or New Arithmetic, and his First Lessons in Algebra; he is not a finished Arithmetician, nor a finished Algebraist. What is to be done in order to finish him? "Put him into Davies' University Arithmetic, and Davies' Bourdon," many will say. But whereabouts in these higher books must we put him? Must he go over the whole of each from the beginning? This would be unnecessary, for he already knows much that is taught in these books, with some variations of phraseology and illustration; and a waste of precious time would thus be the consequence. Shall we confine him to those portions of the higher work which are supplementary to the lower, giving him credit for what he has already achieved! This cannot be done, because these supplementary portions are interwoven with the system of the higher work, and cannot be disentangled from the context. There is in fact no way in which the two books can be used, with the greatest advantage, by the same pupil.

It is entirely practicable to treat all the subjects which are commonly assigned to the lower Arithmetic or Algebra, with all necessary and desirable completeness, and in a manner which shall adapt them fully to the comprehension of pupils of the age at which they are, by judicious instructors, usually put to the study of these sciences. And this will not require that complication of preambles, inferences, analyses, and distinctions without difference, in which the popular Arithmetics especially abound, sometimes (as will hereafter be shown) only "darkening

counsel by words without knowledge ;" nor any of those puerilities, much indulged in, which tend only to disgust the pupil by seeming to presuppose that he knows nothing, and has very little sense.

The first books of Arithmetic and Algebra, if two on each of these subjects must be written, will answer all the purposes of an ordinary education in these sciences. When a higher course in either is to be accomplished, let it ascend from the stage already attained ; let the higher book be complete in itself, but let it be a *continuation of the lower ;* let it be printed from the same plates with the lower, as far as they go, that not even the typography or paging may be changed. When the pupil has learned any thing of the science, let it be once for all ; and when he has associated his knowledge with the parts of his book in which he learned it, as a means of ready recollection and reference let the association remain undisturbed.

When a science is presented in accurate definitions and explicit verbal statements in general, so that the pupil may be required to *study* and *understand* words and sentences which have a precise, clear, and unambiguous meaning, the difficulties of its acquisition are less than is commonly supposed; and it is the want of these very qualities of style, in connection with bad arrangement, in many popular works, that renders so much of explanation in the works themselves, and so much of what is called *teaching* afterwards, necessary to get the student through them. Numerous instances of the deficiency here alleged might be given from almost any of the current Arithmetics and Algebras ; but let a few be taken from Doctor Davies rather than from any other, because this eminent writer has published works on the Grammar of Arithmetic and the Logic of Mathematics, as well as a Mathematical Dictionary ; is himself not averse to criticism ; and his ever-changing, ever-multiplying series of books assumes to be regarded as constituting "Our National

System of Mathematics." I quote first from his New Arithmetic.

"If two or more denominate numbers, having different units, are connected together, forming a single number, such is called a *compound* denominate number. Thus, 3 yards, 2 feet, and 6 inches is a compound denominate number." Where is the *single number* here formed by other numbers having different units? Such a thing is inconceivable; a single number must of necessity express units which, so far as they are denominated at all, are all of one kind.

"Cancellation is a method of shortening Arithmetical operations by omitting or cancelling common factors." From this definition it might be inferred that cancellation is used to shorten any Arithmetical operation, whereas it is applicable only to Division.

"Reduction is the operation of changing the fractional unit without altering the value of the fraction." This sentence stands in the form of a definition of *Reduction* in general; and, in this sense, it is certainly a singular definition. The Doctor is, I believe, inventor of the phrase "fractional unit," of which he makes a very frequent use. Whether there is not an incoherency between the two terms, thus combined, sufficient to preclude the use of the expression, we need not stop to consider; it is enough that such a complexity is not needed in the nomenclature of the science.

"Addition of Fractions is the operation of finding the number of fractional units in two or more fractions." This language can have no definite meaning, inasmuch as it does not ascertain the magnitude of these so-called *fractional units;* and only serves to mystify a subject which needs no definition at all.

"A Vulgar Fraction is one in which the unit is divided into any number of equal parts. A Decimal Fraction is one in which the unit is divided according to the scale of tens." According

to these definitions, $\frac{3}{10}$ is both a *vulgar* and a *decimal* fraction. The second definition does not therefore distinguish a *decimal* fraction. The terms *vulgar* and *decimal*, in contradistinction to each other, refer to the different ways of *denoting* fractions. Any fraction denoted by a numerator and denominator is a vulgar fraction; *tenths*, *hundredths*, etc., when denoted without writing down the denominator, and only then, are *decimal* fractions.

Many other examples of the indefinite and obscure use of language might be taken from Davies', and many other Arithmetics. Davies' Bourdon shall next furnish a few instances of exceptionable definitions.

"Quantity is a general term, applicable to every thing which can be increased or diminished, and measured." The terms "increased and diminished" are too vague in signification to be of any use in this definition: we properly speak of the increase and decrease of *knowledge*, *virtue*, *vice*, etc., and yet these are not quantities. But the definition says that quantity can be increased or diminished, and *measured;* yes, and any thing that can be *measured* is quantity, whether it can be increased and diminished or not. The proper definition is, Quantity is any thing which admits of being measured, so as to be expressed in units.

"Subtraction, in Algebra, is the operation for finding the simplest expression for the difference between two algebraic quantities." This is altogether insufficient. The difference between two algebraic quantities, as found by algebraic subtraction, is ambiguous, since either quantity may be subtracted from the other, and the results will not be the same. If the term *difference* be used at all in the definition, it must be limited to mean the quantity which must be added to the one subtracted to produce that from which the subtraction is made. When the definition is thus made complete and explicit, it leads at once to the manner of performing the operation.

"Multiplication, in Algebra, is the operation of finding the

product of two algebraic quantities." This definition is not incorrect, but it is almost useless from want of explicitness. When the definition of Algebraic Multiplication is made to show that, with a *positive* multiplier, the multiplicand is to be repeatedly *added*, but, with a *negative* multiplier, is to be repeatedly *subtracted*, we have the whole subject before us, with a clue to the *sign* which must be prefixed to the product in any given case.

"An algebraic fraction is an expression of one or more equal parts of unity." This is a proper definition of an arithmetical fraction, in which the two terms are integral numbers. In Algebra the case is different: the two literal terms of the fraction do not *express* integral numbers, nor do we consider them as necessarily representing integral numbers; we in fact operate on these terms in a merely symbolical manner, without the consideration of numerical values at all. An algebraic fraction can only be regarded as representing the quotient of its numerator divided by its denominator, whatever quantities or magnitudes may be symbolized by these terms. The theory of Algebraic fractions should be made independent of the conception of integral numerical numerators and denominators; it should possess all the generality of the algebraic symbols themselves; and it is perfectly easy to establish this theory on this general basis.

"The Ratio of one quantity to another, is the quotient which arises from dividing the second by the first." This is a false definition of ratio: the first quantity must be divided by the second, to find the ratio of the first to the second. In his "Logic and Utility of Mathematics," the Doctor has gone into an elaborate defence of this his method of treating ratios: his reasonings on the subject have been answered in an Appendix to my Elements of Geometry, which the members of this Convention, if they please, may read at their leisure.

These strictures need be pursued no farther; though it would be easy to multiply instances of such inaccurate or indefinite expositions from the works of the same author, and indeed from almost

any of the current works on elementary mathematics. It is on this account, mainly, that these works require to be taught *viva voce;* and hence the almost universal adoption of the blackboard, oral system, with its accompanying "noise and confusion," and the pernicious result to the pupil of leading him to place on a personal teacher that reliance which, for real acquisition, must, after all, be on *himself*, and the text-book before him. There is, doubtless, too much of *teaching* and too little of *studying* in many schools. The only way to correct this evil is to put into the hands of pupils books in which the teacher, in his office of superintendent and examiner chiefly, may *justly* and *steadfastly* require them to learn arithmetic, or algebra, or whatever may be the science in requisition.

The next subject in the mathematical course, and the last on which I shall remark, is Geometry. This science has descended to us from the Greek mathematicians in the celebrated treatise known as Euclid's Elements; respecting which Professor Playfair, of the College of Edinburgh, in the year 1813, says: "It is a remarkable fact, in the history of science, that the oldest book of Elementary Geometry is still considered as the best." Professor Young, in 1827, testifies that in Great Britain an almost universal preference is given to Euclid. Both of these eminent mathematicians wrote subsequently to the publication of Legendre's work, which, in the translation of Sir David Brewster, is the basis of Davies' Legendre, Loomis's Geometry, and one or two other works of less note, written in this country. These American editions of Legendre are the principal text-books on Geometry in the United States.

The history of this noble science possesses considerable interest in a merely logical or metaphysical point of view. Human reason has here labored to connect together a vast system of absolute truths by the evidence of absolute intuition. Unfettered

by any thing but its own laws of operation, it has gone to work to determine and define those first truths in the science of extension and figure from which all its other truths may be logically deduced. These first truths, presented under the form of definitions, are of the nature of postulates, or facts which are evidently possible. These having been assumed, all subsequently asserted truths must be shown to arise, immediately or remotely, by necessary inference, out of them; and the logic of the science is complete only when this has been accomplished.

Has this ever been accomplished? In other words, have we a true science of Geometry? This may seem to many a very needless inquiry, when it is considered that in the current standard treatises, and in the customary teaching of Geometry, no allusion is made to any missing link in the concatenation of principles presented—no notice taken of any weakness whatever in any part of the foundation on which the science is built. But if we ask the great masters of this science, they will tell us that the theorems respecting *parallel lines* have not been satisfactorily demonstrated: not that these theorems are not *true*, but that they have not been *logically proved.*

The illustrious author—the author to us, at least—of this science, gave a *definition* of parallels which almost every succeeding writer, to a very recent period, has adopted, and which I take from Davies' Legendre.

"Two straight lines are said to be *parallel*, when, being situated in the same plane, they cannot meet, how far soever, either way, both of them be produced."

From this definition Euclid wished to derive the theorem that the alternate angles which a straight line makes with two parallel straight lines are equal to each other. Finding nothing in the definition itself from which this truth could be inferred, he premises, as an *axiom*, the proposition, substantially, which stands as the twenty-first theorem in Davies' Legendre: "If two

straight lines meet a third line, making the sum of the two interior angles on the same side less than two right angles, the two lines will meet if sufficiently produced." This is far from being *axiomatic;* and the logical imperfection of Euclid's method of arriving at the truth to be established is clearly and confessedly manifest.

Legendre attempts the demonstration of a lemma, from which he infers the truth of Euclid's axiom above stated. His demonstration, according to his own account of it, "has not the same character of rigorousness with the other demonstrations of elementary Geometry." He thus implicitly confesses his inability to remove the defect in Euclid's method.

Very great efforts have been made by foreign geometers to establish an unexceptionable theory of parallel lines; and the candid admission of such writers as Legendre, Playfair, and Young is, that the object has not been accomplished in a manner which is entirely rigorous, and, at the same time, suitable for elementary instruction. Other definitions than Euclid's have been tried, but without any more satisfactory results. Metaphysically, here is an interesting state of things—a mathematical theory received as absolutely *true*, which could not be logically *proved.*

Our own most popular geometrical writers have blinked the difficulties of this part of the science altogether; but they agree with each other as to the best method of cutting the knot which they have declined to untie. This method—first employed, I think, by Playfair, yet not fully approved by him—is very specious, but is nothing more. The main difficulty consists in passing from Euclid's definition of parallels, which is the common one, to the conclusion that when a straight line meets two parallel straight lines, the alternate angles will be equal; or to any positive conclusion whatever in the doctrine of parallels. In Davies' Legendre, and in Loomis's Geometry, we find this pas-

sage attempted through the medium of this axiom: "Through the same point only one straight line can be drawn which shall be parallel to a given line." This is true, but it is not *self-evident* from these authors' definition of parallels. The science of Geometry, let it be borne in mind, must be derived entirely from the definitions in Geometry—it has no other source—the axioms themselves must follow from the definitions, just as corollaries follow from theorems. What does our authors' definition of parallels contain? Nothing but the conception of two straight lines, in the same plane, and infinitely prolonged both ways, if you please. without meeting each other. How does the axiom above stated follow from this definition? The line A cannot meet the line B—these are the parallels; the line C meets the line A, these lines passing through the same point; therefore the line C also meets the line B. This inference, to my apprehension, is a very plain *non-sequitur;* and this is the precise inference contained in the aforementioned axiom. The impassable gulf still remains between the definition and the subsequent doctrine of parallels.

Doctor Davies might have profited by following his "model and guide," Legendre, more closely in some things than he has seen proper to do. This "first geometer of Europe," as he has been called, speaking of the employment of this identical axiom by Leslie, says: "There is only this difference between Mr. Leslie's method and that of Euclid, that the ancient geometer does not dissemble the difficulty, but presents it, on the contrary, in all its breadth, and requires to have that granted which he cannot prove; while the modern geometer envelops the difficulty in a shadow of demonstration which, though doubtless it has seduced himself, is certainly anything but rigorous."

In the nature of things it must be possible, and I will presently undertake to show that it is also practicable, to establish an unexceptionable theory of parallelism. It must be derived from

the definitions of a straight line, of an angle, and of parallel lines. Let us briefly consider some of the various definitions that have been given of the straight line and angle.

Euclid's definition of a straight line, in the translation of Playfair, is, that "A straight line lies evenly between its extreme points;" which, as the translator remarks, is obviously faulty, the word *evenly* standing as much in need of definition as the word *straight*, which it is intended to define.

Legendre has it that "A straight line is the shortest distance from one point to another;" and this definition has been retained in all the various editions of his work, with the exception of Davies' last edition. This appears to me to be an inference from a more primary conception of a straight line, and therefore not properly its definition.

Various other definitions have been given, which it is not necessary to notice. That which appears to be the true one, as embodying our most radical and simple conception of straightness, has been adopted, sucessively, by Hutton, Hayward, Pierce, Hackley, Davies in his revised edition, and Perkins in his new Geometry. It is this: A straight line is one which has the *same direction* throughout its whole extent.

The definitions of an angle have been equally various with those of a straight line. Euclid defines it to be "the inclination of two straight lines to one another, which meet together, but are not in the same straight line." This has passed, without objection, through the hands of his distinguished commentators, Simpson and Playfair. Young objects to it—justly, I think—that it is very vague, conveying but an indistinct notion of angular magnitude; and himself defines an angle as the opening between two straight lines which meet each other.

Legendre, as translated by Brewster, says, "When two straight lines meet together, the quantity, greater or less, by which they are separated from each other in regard to their position, is called

an angle." This definition very properly regards an angle as a *quantity*, which is a position that has sometimes been disputed. But the definition itself is greatly wanting in simplicity, and has been discarded by all who have ventured any alterations in the work of the distinguished Frenchman.

In the former editions of Davies' Legendre, and Loomis's Geometry, we have a kind of alternative definition, embracing Euclid's and Young's; thus: "When two straight lines meet each other, their *inclination* or *opening* is called an angle." This seems calculated to produce a confusion of ideas: the terms *inclination* and *opening* express very different things to my apprehension; so do the terms *opening* and *meeting*.

Pierce says that "An angle is the difference of direction of two straight lines meeting or crossing each other;" and the same definition has been adopted by Perkins in his new Geometry. This does not reach the points of utmost simplicity and clearness.

In his last edition, Davies says, "A plane angle is the portion of a plane included between two straight lines meeting at a common point." This is certainly a false definition; an angle is not a portion of a plane; nor can a portion of a plane be included between two straight lines. For suppose an angle with sides of any determinate length; produce the sides without altering their relative position; now, if there was a portion of a plane included between the sides at first, it was increased when the sides were produced; and the angle was thus increased; which is false.

The best definition appears to be this: A plane angle, or simply an angle, is the *divergence* of two straight lines proceeding from the same point. This seems to be perfectly simple and perspicuous; and this term *divergence* may be used, in like manner, in defining every kind of angle. A *diedral* angle is the divergence of two planes proceeding from the straight line of their common intersection; a *polyhedral* angle is the divergence of sev-

eral plane angles proceeding from a common vertex; a *spherical* angle is the divergence of two arcs of great circles of the same sphere.

Without considering all the definitions that have been given of parallel lines, I proceed to that which is certainly the true one:

"Parallel straight lines are such as have the *same direction* with each other—being, in order to this, in the *same plane*."

This definition, in substance, appears to have originated with Mr. Hayward, formerly Professor of Mathematics and Natural Philosophy in Harvard University. It was given in his Elements of Geometry, published in 1829; in the work of Professor Pierce, of the same University, in 1837; in Hackley's Geometry, in 1847; and was adopted by Perkins, in his new Geometry, published last year. Davies and Loomis still adhere to Euclid's definition.

Professor Hayward introduces this new definition with the following remarks, which have always appeared to me to be exceedingly just and appropriate: "The above definition of parallel lines is adopted because it is believed to characterize the relation of the parallels to each other, more precisely than those definitions which make the parallelism of the lines consist in their *not meeting* or their *being throughout at the same distance from each other*. Parallel lines are throughout at the same distance from each other, and cannot meet. These are truths which result from the property or rather the *relation* of parallelism; that is, from their having the same direction. *This identity of direction* is what constitutes the parallelism of the lines. And this notion of parallels should be, at first, presented to the contemplation of the learner; for this is the simple principle from which result all those propositions that make up what is called the doctrine of parallel lines." And again, speaking of the works which adopt Euclid's definition of parallels, he says: "It is hardly credible that the authors themselves, in using parallel

lines in the various demonstrations in which they occur, usually think of them as *not meeting.* They contemplate them merely as having the same direction, and mentally derive their results from this property. This is certainly true of those who read their books."

We may now untie the Gordian knot which has been cut by so many Alexanders: we may pass from the *definition* to the *doctrine* of parallelism. A straight line has the same *direction* throughout its whole extent—parallel straight lines have the *same direction with each other*—an angle is the *divergence* of two straight lines proceeding from the same point. Two straight lines having the same direction with each other, diverge *equally* from any straight line meeting them; that is, the opposite *exterior* and *interior* angles are equal; and since vertical angles are equal, it follows that the *alternate* angles are also equal. From these principles the whole theory of parallelism, for straight lines and planes, may be logically and easily deduced.

With this direct, simple, and conclusive method, compare the indirect, complex, and inconclusive method which presents, *first*, a Definition expressive only of a *negative* property, connected with the notion of *infinity*, (Davies' 16th;) *secondly*, an Axiom which does not follow from this definition, (Davies' 13th;) *thirdly*, a Theorem sought to be demonstrated through the medium of this axiom and a *reductio ad absurdum*, (Davies' 20th.)

There has been another formidable difficulty in the logic of Geometry. This consists in the establishment of Proportion among incommensurable magnitudes, or those which, from having no finite common measure, are not to one another in the proportion of any *finite numbers.* Euclid is generally admitted to have overcome this difficulty by proceeding upon a definition which does not attribute numerical values to the terms of the proportion; but his method is so obscure as to be totally un-

fit for the purposes of instruction; it has not, therefore, been followed in any modern treatise. Legendre does not treat of the general theory of Proportion, but for the principles of his theory refers the student to the common treatises on Arithmetic and Algebra. In the current editions of Legendre, the arithmetical or algebraical theory of Proportion has very properly been inserted; as in the second Book of Davies' Legendre.

This recourse to Arithmetic or Algebra for the theory of Proportion, is doubtless the only possible alternative, on the abandonment of Euclid's method; but Legendre and his followers are inconsistent with themselves in some of their procedures under this theory, as shall presently be shown.

Proportion consists in an equality of ratios. Euclid defines ratio to be "a mutual relation of two magnitudes, of the same kind, to one another, in respect of quantity;" but he does not show how the quantitive relation of two magnitudes to each other is to be expressed or conceived. It can be expressed or conceived only through the medium of numbers representing those magnitudes; and it is just this want of numerical values in the terms of a proportion that has produced the obscurity of Euclid's theory, in which he makes no direct comparison of the terms themselves with one another, but compares certain equimultiples of the terms. This was his expedient for bringing incommensurable magnitudes, as well as commensurable ones, under the dominion of proportion.

Now, in the theory of Proportion, as expounded in Arithmetic and Algebra, the terms must be considered *numerically;* and the predication of ratio between any two magnitudes, according to this theory, is necessarily the predication of relative numerical values in the terms of the ratio. There are no magnitudes, therefore, which can be made the subjects of Proportion that are not expressible, actually or imaginably, by numbers. This is asserted by Legendre. Speaking of the equality between

the product of the extremes and that of the means, he says: "This truth is indisputable, so far as concerns numbers: it is equally so in regard to magnitudes of any kinds, provided they are expressed, or imagined to be expressed, in numbers; and this we are at all times entitled to imagine."

The inconsistency of Legendre and his followers will now be made apparent. They commit themselves to a doctrine of Proportion in which the definition of *ratio* is founded on the conception of relative *numerical values* in the terms of the ratio, and afterwards theorize about ratios between magnitudes to which they do not allow the possibility of relative numerical values at all. This inconsistency is fallen into whenever they employ *reductio ad absurdum* in treating of the ratio between two lines supposed to be incommensurable, for, in their view, the lines have no numerical representatives—through which alone a quantitive relation, or ratio, between two magnitudes is predicable or conceivable. This is to affirm a ratio, and, at the same time, to suppose the want of that numerical relation in which ratio consists. Legendre, in the extract already given, affirms that proportional magnitudes may always be imagined to be expressed in numbers: by adhering to this view, he and his followers might have avoided that frequent use of the *reductio ad absurdum* which, notwithstanding the commendation of Sir David Brewster, unnecessarily and inconsistently complicates so many of their demonstrations in proportion.

But how can incommensurable magnitudes, as they are called, —that is, magnitudes having no *finite* common measure,—be imagined to be expressed in numbers? The answer is, For the purpose of demonstration, they may be supposed to have an *infinitesimal* common measure, or one which is diminished *without limit.* The demonstrations depending on this supposition will be perfectly simple and rigorous; and they will be in consistency with the theory of Proportion which Legendre adopts, while his

method of *reductio ad absurdum*, for incommensurables, is a manifest departure from that theory.

The consideration of infinitely small quantities will thus sometimes enter into demonstrations concerning the proportionality of magnitudes, as it has long done into those concerning certain properties of the circle and sphere. Newton and Leibnitz employed this instrumentality in building one of the higher departments of mathematical science: humbler men may employ it in repairing one of the lower.

The logic of Geometry, as it stands in the most popular treatises, in reference to the theories of parallelism and proportionality, may thus be improved. There are also other improvements which may be effected in the current treatises on this science. It is practicable to attain greater succinctness and perspecuity of verbal statement in many places—to introduce some new principles which have important applications—to improve some old demonstrations—greatly to diminish the number of indirect demonstrations—and, especially, to adopt a more systematic arrangement of the different parts of the science.

In regard to the method of Geometry, a few remarks may be offered. The ancient Geometry, as we have it in Euclid, was by the method of diagrams addressed to the eye, but designed only to suggest to the imagination the abstract figures or forms of the science, which can have no true prototypes among things visible. The demonstrations were conducted with constant reference to these imagined figures, and the study of Geometry was thus, at the same time, an exercise of the reason and of the imagination. The modern Geometry tends more and more to turn away from the conception of figured magnitudes, and to run into merely algebraic operations. The ancient method is, by the most enlightened minds, regarded as incomparably the best for the purposes of mental culture, while the modern is admitted to possess supe-

rior advantages for mere necessary investigation. Every one who has had the good fortune to form his geometrical taste upon the model of Euclid, must feel that this science has been despoiled of much of its beauty, and deprived of much of its utility as an instrument of education, in being subjected to the form in which we now find it in some of the more recent works in which it is presented to the youth of our country.